Conjugacy of orientation preserving Morse-Smale diffeomorphisms graphs.

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In the present paper we consider preserving orientation Morse-Smale diffeomorphisms on surfaces. We will consider class $MS(M^2)$ o.p. Morse-Smale diffeomorfisms on M^2 – smooth closed connected orientamle 2-dimensional manifold. Using the combinatorics theory and theory of knots and links we will show, that conjugacy of two graphs G, G' built on the two mapping f, f' is enough to say, that f, f' are conjugated.

Definition 1 (Orientable heteroclinic intersections)**.** *Let $f \in MS(M^2)$, σ_i, σ_j — saddle points of diffeomorphism f, such that $W^s_{\sigma_i} \cap W^u_{\sigma_j} \neq \emptyset$. For any heteroclinic point $x \in W^s_{\sigma_i} \cap W^u_{\sigma_j}$ define an ordered pair of vectors $(\vec{v}^u_x, \vec{v}^s_x)$, where:*

- $\vec{v}^u_x$ *— the tangent vector to the unstable manifold of the point σ_j at the point x;*
- $\vec{v}^s_x$ *— the tangent vector to the stable manifold of the point σ_i at the point x.*

*Heteroclinics intersections of diffeomorphism f called **orientable**, if an ordered pairs of vectors $(\vec{v}^u_x, \vec{v}^s_x)$ set the same orientation of the bearing surface M^2. Otherwise heteroclinic intersection is called non-orientable.*

Using , introduced by S. Smale [3], partial orderliness relation $\prec$ we shatter Σ_f (the set of periodic mapping orbits of f) on a subset Σ_i as follows:

- Σ_0 — the set of all stable orbits ω;
- Σ_1 — the set of all saddle orbits σ_i, such $W^u_{\sigma_i}$ does not contain heteroclinic points.
- Σ_2 — the set of remaining saddle orbits of the system.
- Σ_3 — the set of all unstable orbits α

Sets Σ_i is ordered in the following way:

$$\Sigma_0 \prec \Sigma_1 \prec \Sigma_2 \prec \Sigma_3$$

Let G is class of maps f, such $f : M^2 \to M^2$ - orientation preserving Morse-Smale diffeomorphism on smooth closed connected orientable 2-dimensional manifold.

In papers [1] and [2] shown, that $beh(f) = 1$ (have finite number of heteroclinic orbits). In present paper we will show that the classification is reduced to the combinatorial problem of distinguishing graphs.

Set that $\mathcal{A}_f = \Sigma_0 \cup W^u_{\Sigma_1}$ — is the attractor of the system, then $\mathcal{R}_f = \Sigma_3 \cup W^s_{\Sigma_2}$ — repeller of the system. In paper [4] shown, that the chosen sets are the attractor and repeller of the system respectively.

Set that $V_f = M^2 \backslash (\mathcal{A}_f \cup \mathcal{R}_f)$, then make factorization we get the quotient spase $\hat{V}_f = V_f/f$. Let n — the number of tori of which is consist the quotient space $\hat{V}_f$ [], i. e. $\hat{V}_f = \bigsqcup\limits_{i=1}^{n} \hat{V}_f^i$.

Let is introduce the canonical projection $p_f : V_f \to \hat{V}_f$, such $p_f(L^u) = \hat{L}^u$, $p_f(L^s) = \hat{L}^s$, where L^u, L^s – families of unstable and stable separatrices.

Also for any $f \in G$ exist scheme $S_f = (\hat{V}_f, \hat{L}^u, \hat{L}^s)$. From paper [] follows, that $f, f' \in G$ are topologically conjugatd $\Leftrightarrow$ S_f is equivalent to $S_{f'}$

Now we can put each scheme in accordance with the graph. We define the graph P as follows:

$\bigsqcup\limits_{i=1}^{n} B_i(\hat{V}_f^i, \hat{L}_i^s, \hat{L}_i^u)$ – the set of vertices of a graph to which we add one vertex for each separatrix of $\hat{L}^s, \hat{L}^u$ and one vertex ϱ_i, that denoting belonging to the torus.

$E(\hat{V}_f, \hat{L}_i^s, \hat{L}_i^u)$ – the set containing edges of several types:

- Edges corresponding to connectivity components of $\hat{V}_i \setminus W_i^s$, that connect vertices corresponding to separatrices $\hat{L}_i^s$.
- Edges corresponding to connectivity components of $\hat{V}_i \setminus W_i^u$, that connect vertices corresponding to separatrices $\hat{L}_i^u$.
- Edges corresponding to connectivity components of $W^s_\sigma \setminus \sigma(\sigma \in \Sigma_1)$, that connect vertices corresponding to separatrices $\hat{L}^s$, which can also belong to different tori.

- Edges corresponding to connectivity components of $W^u_\sigma \setminus \sigma$(where $\sigma \in \Sigma_2$), that connect vertices corresponding to separatrices $\hat{L}^s$, which can also belong to different tori.
- Edges that connect all vertices of the set $B_i(\hat{V}^i_f, \hat{L}^s_i, \hat{L}^u_i)$ and vertex ϱ_i.

Theorem 1. *If f topologically conjugated to f', then graphs P and P' are isomorphic*

References

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