On dual description of the $OSp(N|2m)$ sigma models

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Motivation

- The integrability-preserving deformations of $O(N)$ sigma models are known to admit the dual description in terms of a coupled theory of bosons and Dirac fermions with exponential interactions of the Toda type (Fateev'04, Litvinov, Spodyneiko'18).
- On the other hand, there are known examples of the integrable superstring theories, such as type IIB $\mathrm{AdS}_5 \times \mathrm{S}^5$ (dual to $\mathcal{N} = 4$ SYM) and others, which also have integrable deformations.
- Our strategic goal is to build a similar dual description for the deformed $\mathrm{AdS}_5 \times \mathrm{S}^5$ type IIB superstring and, possibly, other models.
- The building of such a dual description for the superstring theory requires solving three major problems:
  1. Incorporate the fermionic degrees of freedom into the construction of dual theory.
  2. Adapt the whole construction to describe the sigma models with non-compact target space.
  3. The superstring theory possesses the reparametrization symmetry and requires gauge fixing, which makes us include this symmetry into the dual description.
- In the present work we address the first problem generalizing the dual description of the deformed $O(N)$ sigma models to account for the $OSp(N|2m)$ sigma models.

The undeformed $OSp(N|2m)$ sigma model

- The $OSp(N|2m)$ sigma model is given by the symmetric space sigma model on the supercoset

$$\frac{OSp(N|2m)}{OSp(N-1|2m)} \, .$$

- The action for the supergroup-valued field $g \in OSp(N|2m)$ is

$$\mathcal{S}_0 = -\frac{R^2}{2} \int d^2x \, \mathrm{STr}[J_+ P J_-] \, ,$$

where $J_\pm = g^{-1}\partial_\pm g$ takes values in the Grassmann envelope of the Lie superalgebra $\mathfrak{osp}(N|2m;\mathbb{R})$ and STr is the invariant bilinear form.

- We are considering the symmetric space with the $\mathbb{Z}_2$ grading

$$\mathfrak{g} \equiv \mathfrak{osp}(N|2m;\mathbb{R}) = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \, , \quad \mathfrak{g}^{(0)} = \mathfrak{osp}(N-1|2m;\mathbb{R})$$

and P being the projector onto the grade 1 subspace.

- This model is quantum integrable and has the following rational S-matrix (Saleur, Wehefritz-Kaufmann'01)

$$\check{S}^{j_2 j_1}_{i_1 i_2}(\theta) = \sigma_1(\theta) E^{j_2 j_1}_{i_1 i_2} + \sigma_2(\theta) P^{j_2 j_1}_{i_1 i_2} + \sigma_3(\theta) I^{j_2 j_1}_{i_1 i_2} \, .$$

The coefficients in front of the tensor structures are connected as follows

$$\sigma_1(\theta) = -\frac{2i\pi}{(N-2m-2)(i\pi-\theta)}\sigma_2(\theta) \, , \quad \sigma_3(\theta) = -\frac{2i\pi}{(N-2m-2)\theta}\sigma_2(\theta) \, .$$

Trigonometric OSp(N|2m) R-matrix

- Besides rational solution, the Yang-Baxter equation

$$\check{R}^{k_2k_1}_{i_1i_2}(\mu)\check{R}^{k_3j_1}_{k_1i_3}(\mu+\rho)\check{R}^{j_3j_2}_{k_2k_3}(\rho)=\check{R}^{k_3k_2}_{i_2i_3}(\mu)\check{R}^{j_3k_1}_{i_1k_3}(\mu+\rho)\check{R}^{j_2j_1}_{k_1k_2}(\rho)$$

  has the trigonometric solution (Bazhanov, Shadrikov'87) with the parameter q.
- Introducing the parametrization

$$q=e^{2i\pi\lambda}\,,\quad \mu=(N-2m-2)\lambda\theta\,,$$

  we observe that for $\lambda=0$ it is consistent with the rational limit and in the special point $\lambda=\frac{1}{2}$ the R-matrix demonstrates an interesting behaviour. It becomes proportional to the S-matrix, corresponding to the scattering of $\frac{N}{2}$ Dirac fermions and m superghost particles in the case of even N and the same plus one boson in the case of odd N.
- The $O(3)$ example with $N=3$, $m=0$ at $\lambda=\frac{1}{2}$:

$$\frac{\check{R}^{j_2j_1}_{i_1i_2}}{\check{R}^{22}_{22}}=\begin{pmatrix}\begin{pmatrix}-1&0&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&-1\\0&0&0\\0&0&0\end{pmatrix}\\\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}\\\begin{pmatrix}0&0&0\\0&0&0\\-1&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}&\begin{pmatrix}0&0&0\\0&0&0\\0&0&-1\end{pmatrix}\end{pmatrix}+\mathcal{O}\left(\lambda-\frac{1}{2}\right)\,.$$

Building of the dual model

- In the work (Fateev, Onofri, Zamolodchikov'93) there was studied the dual description of the sigma model with the metric ($\lambda=\nu+\mathcal{O}(\nu^2)$)

$$ds^2=\frac{\kappa}{\nu}\left(\frac{dr^2}{(1-r^2)(1-\kappa^2r^2)}+\frac{1-r^2}{1-\kappa^2r^2}d\phi^2\right)$$

  i.e. the special integrable perturbation of the Sine-Liouville theory ($\lambda=\frac{1}{2}-\frac{b^2}{2}+\mathcal{O}(b^4)$)

$$\begin{aligned}\mathcal{L}=&\frac{(\partial_\mu\Phi)^2}{8\pi}+\frac{(\partial_\mu\varphi)^2}{8\pi}-\\&-\frac{m}{4}\left(e^{b\Phi+i\beta\varphi}+e^{b\Phi-i\beta\varphi}+e^{-b\Phi+i\beta\varphi}+e^{-b\Phi-i\beta\varphi}\right)-\\&-\frac{m^2}{32\pi b^2}\left(e^{2b\Phi}-2+e^{-2b\Phi}\right),\quad \beta=\sqrt{1+b^2}\,.\end{aligned}$$

  Th sigma model coupling constant in the regime $b\to\infty$ is $\nu=\frac{2}{b^2}+\mathcal{O}\left(\frac{1}{b^4}\right)$.

- Guiding principles to look for the dual description (Litvinov, Spodyneiko'18)
  1. The theory with the S-matrix as above has to be renormalizable (at least 1-loop).
  2. The dual theory is found as an integrable perturbation from the special point of the S-matrix and is determined by the set of screening charges, which commute with the integrals of motion in the leading order in the mass parameter

$$\left[I_k^{\text{free}},\int e^{(\boldsymbol{\alpha}_r,\phi)}dz\right]=0\,.$$

  3. Our model is an integrable deformation of the CFT, based on the coset $\frac{\widehat{\mathfrak{osp}}(N|2m)_{\mathcal{W}}}{\widehat{\mathfrak{osp}}(N-1|2m)_{\mathcal{W}}}$.

The Yang-Baxter deformation of the $OSp(N|2m)$ sigma model

- The action for the Yang-Baxter deformed model is (Klimcik'02,Delduc'13)

$$\mathcal{S}_\eta = \int d^2x\, \mathcal{L}_\eta = -\frac{\eta}{2\nu} \int d^2x\ \mathrm{STr}[J_+ P \frac{1}{1-\eta \mathcal{R}_g P} J_-]\,,$$

where η is the deformation parameter and ν is the sigma model coupling.

- The operator $\mathcal{R}_g$ is defined in terms of an operator $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$ through

$$\mathcal{R}_g = \mathrm{Ad}_g^{-1}\, \mathcal{R}\, \mathrm{Ad}_g\,,$$

with $\mathcal{R}$ an antisymmetric solution of the (non-split) modified classical Yang-Baxter equation

$$[\mathcal{R}X, \mathcal{R}Y] - \mathcal{R}([X, \mathcal{R}Y] + [\mathcal{R}X, Y]) = [X, Y]\,,$$
$$\mathrm{STr}[X(\mathcal{R}Y)] = -\mathrm{STr}[(\mathcal{R}X)Y]\,, \quad X, Y \in \mathfrak{g}\,.$$

- In terms of coordinates on the target superspace

$$\mathcal{L}_\eta = (G_{MN}(z) + B_{MN}(z))\, \partial_+ z^N \partial_- z^M\,, \quad z^M = (x^\mu, \psi^\alpha)\,,$$

where $G_{MN} = (-1)^{MN} G_{NM}$ and $B_{MN} = -(-1)^{MN} B_{NM}$.

- We explicitly calculated $G_{MN}(z)$ and $B_{MN}(z)$ in the range of parameters $N = 1, \ldots, 8$ and $m = 1, 2, 3$.

Ricci flow

- Substituting the metric and Kalb-Ramond field of the deformed $OSp(N|2m)$ sigma model for $m = 1$ with $N = 1, \ldots, 6$ into the Ricci flow equation
$$R_{MN} + \frac{d}{dt} E_{MN} + (\mathcal{L}_Z E)_{MN} + (dY)_{MN} = 0 \,, \quad E_{MN} = G_{MN} + B_{MN} \,.$$
we indeed find ($t \sim \log \Lambda_{UV}$)
$$\frac{d\nu}{dt} = 0 \,, \quad \frac{d\eta}{dt} = -\nu(N - 2m - 2)(1 + \eta^2) \,.$$
which is the natural expectation for general N and m. It agrees with the known result for $m = 0$ (Squellari'14, Litvinov, Spodyneiko'18).
- Taking $\nu = \eta R^{-2}$ with $\eta \to 0$, we find the RG flow in the undeformed limit
$$\frac{dR^2}{dt} = -(N - 2m - 2) \,.$$
- Solving the renormalisation group flow equations for real η we find cyclic solutions. This motivates us to consider the analytically-continued regime
$$\nu \to i\nu \,, \quad \eta \to i\kappa \,,$$
in which we have ancient solutions. In this regime the solution is
$$\nu = \text{constant} \,, \quad \kappa = -\tanh\left(\nu(N - 2m - 2)t\right) \,.$$
- Therefore the model in question is asymptotically free in the UV for $N - 2m > 2$. From now on we will concentrate on the simplest case of this type, i.e. $N = 5$ and $m = 1$ or $OSp(5|2)$.

$OSp(N|2m)$ action from $O(N+2m)$ action

- Although the general form of this trick is known to us, for conciseness let us consider the case $N = 2n+1$ and $m = 1$. The simplest way to write the deformed $O(2n+1)/O(2n)$ action is to use "stereographic" coordinates

$$ds^2 = \sum_{k=1}^{n} \frac{\kappa_k}{\nu} \frac{dz_k d\bar{z}_k}{(1+z_k\bar{z}_k)^2 \left(1 - \kappa_k^2 \left(\frac{1-z_k\bar{z}_k}{1+z_k\bar{z}_k}\right)^2\right)} \, ,.$$

  where

$$\kappa_k = \kappa \prod_{j=1}^{k-1} \left(\frac{1-z_j\bar{z}_j}{1+z_j\bar{z}_j}\right)^2 \, , \quad k = 1, \ldots, n \, .$$

- The transition to different deformations $OSp(N|2)$ action from the $O(N+2)$ is made by the substitution for some z_k

$$z_k \to \frac{\psi}{\sqrt{2}} = \frac{\psi^1 + i\psi^2}{\sqrt{2}} \, , \quad \bar{z}_k \to \frac{\bar{\psi}}{\sqrt{2}} = \frac{\psi^1 - i\psi^2}{\sqrt{2}} \, .$$

  Further we concentrate on the case $k = 1$.

- Also we go back to the "spherical" parametrization of the coordinates z_j

$$z_j = \sqrt{2\frac{1-r_j}{1+r_j}} e^{i\phi_j} \, .$$

The deformed $OSp(5|2)$ sigma model action

- Let us now turn to the specific case $OSp(5|2)$. The deformed sigma model is parametrised by four bosons, ϕ_1, ϕ_2, r_1 and r_2, and a symplectic fermion, ψ^a, where $a = 1, 2$.

- The Lagrangian following from the previous slide is

$$\begin{aligned}
\mathcal{L}_\kappa = \frac{\kappa(1-\kappa^2 r_1^2 + (1+\kappa^2 r_1^2)\psi\cdot\psi)}{\nu(1-\kappa^2 r_1^2)^2}\Big[\frac{\partial_+ r_1 \partial_- r_1}{1-r_1^2} + (1-r_1^2)\partial_+\phi_1\partial_-\phi_1 \\
+ i\kappa r_1(1+\psi\cdot\psi)(\partial_+ r_1\partial_-\phi_1 - \partial_+\phi_1\partial_- r_1)\Big] \\
+ \frac{\kappa r_1^2(1-\kappa^2 r_1^4 r_2^2 + (1+\kappa^2 r_1^4 r_2^2)\psi\cdot\psi)}{\nu(1-\kappa^2 r_1^4 r_2^2)^2}\Big[\frac{\partial_+ r_2\partial_- r_2}{1-r_2^2} + (1-r_2^2)\partial_+\phi_2\partial_-\phi_2 \\
+ i\kappa r_1^2 r_2(1+\psi\cdot\psi)(\partial_+ r_2\partial_-\phi_2 - \partial_+\phi_2\partial_- r_2)\Big] \\
- \frac{\kappa(1-\kappa^2+\frac{1}{2}(1+\kappa^2)\psi\cdot\psi)}{\nu(1-\kappa^2)^2}\left[\partial_+\psi\cdot\partial_-\psi - i\kappa(1+\tfrac{1}{2}\psi\cdot\psi)\partial_+\psi\wedge\partial_-\psi\right],
\end{aligned}$$

where we have introduced the following contractions of the symplectic fermion

$$\chi\cdot\chi' = \epsilon_{ab}\chi^a\chi'^b\,, \quad \chi\wedge\chi' = \delta_{ab}\chi^a\chi'^b\,.$$

UV limit of the deformed OSp(5|2) sigma model

- We are interested in the expansion around the UV fixed point, that is $\kappa = 1$. The specific limit we consider (Litvinov'18) is given by first setting

$$r_1 = \exp(-\epsilon e^{-2x_1}) , \quad r_2 = \tanh x_2 , \quad \psi^a = \epsilon\theta^a , \quad \kappa = 1 - \frac{\epsilon^2}{2} ,$$

  and subsequently expanding around $\epsilon = 0$.

- Introducing the complex fields

$$X_1 = x_1 - i\phi_1 , \qquad X_2 = x_2 - i\phi_2 , \qquad \Theta = \theta^1 - i\theta^2 ,$$

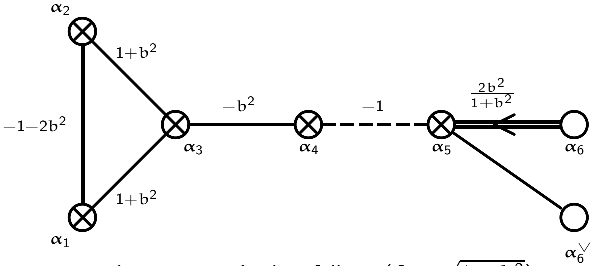
  we find the following expansion

$$\begin{aligned}\mathcal{L}_{\kappa\sim 1} = \frac{1}{\nu} &\left(\partial_+ X_1 \partial_- X_1^* + \partial_+ X_2 \partial_- X_2^* + i(1 - i\Theta\Theta^*)\partial_+\Theta\partial_-\Theta^*\right) \\ &- \frac{\epsilon}{\nu}\Big(\tfrac{1}{2}e^{2x_1}(1 + 2i\Theta\Theta^*)\partial_+ X_1 \partial_- X_1^* + \\ &+ e^{-2x_1+2x_2}\partial_+ X_2 \partial_- X_2^* + e^{-2x_1-2x_2}\partial_+ X_2^* \partial_- X_2\Big) + \mathcal{O}(\epsilon^2) ,\end{aligned}$$

  up to total derivatives.

Screening charges for the deformed $OSp(5|2)$ sigma model

- We propose dual description of $OSP(5|2)$ deformed sigma-model. Its system of screening charges is the following



- The vectors $\boldsymbol{\alpha}_r$ can be parameterized as follows ($\beta = \sqrt{1+b^2}$)

$$\boldsymbol{\alpha}_1 = bE_1 + i\beta e_1\,, \qquad \boldsymbol{\alpha}_2 = -bE_1 + i\beta e_1\,, \qquad \boldsymbol{\alpha}_3 = -bE_2 - i\beta e_1\,,$$

$$\boldsymbol{\alpha}_4 = bE_2 + i\beta e_2\,, \qquad \boldsymbol{\alpha}_5 = \frac{i}{\beta}e_2 + \frac{ib}{\beta}e_3\,, \qquad \boldsymbol{\alpha}_6 = -\frac{2ib}{\beta}e_3\,,$$

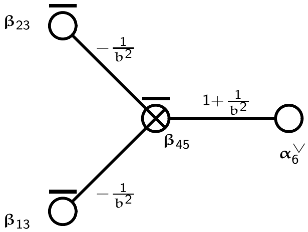
- To restore the sigma model metric in the UV limit, we have to use the fact (Litvinov, Spodyneiko'16) that for the pair of the neighbouring fermionic exponential screenings $e^{(\boldsymbol{\alpha}_1,\phi)}$ and $e^{(\boldsymbol{\alpha}_2,\phi)}$ the dressed screenings

$$(\boldsymbol{\alpha}_{1,2},\partial\phi)e^{(\boldsymbol{\beta}_{12},\phi)}\,, \quad \boldsymbol{\beta}_{12} = \frac{2(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)}{(\boldsymbol{\alpha}_1+\boldsymbol{\alpha}_2)^2}$$

commute with the same system of the integrals of motion.

Metric for the deformed OSp(5|2) sigma model

- By taking the dual screenings we obtain the following system, which includes the dressed screenings



- By choosing $z = x^1 - ix^2$ ($\bar{z} = x^1 + ix^2$) and then conducting Wick rotation $x^2 = ix^0$, we obtain the action in Minkowski signature

$$\mathcal{L} = \frac{1}{2\pi}\left(\sum_{i=1}^{2}(\partial_+\Phi_i)(\partial_-\Phi_i) + \sum_{j=1}^{3}(\partial_+\phi_j)(\partial_-\phi_j)\right) +$$
$$+\Lambda_1\Big(\partial_+(b\Phi_1 + i\beta\phi_1)\,\partial_-(b\Phi_1 - i\beta\phi_1)\,e^{\frac{\Phi_1-\Phi_2}{b}} +$$
$$+\partial_+(b\Phi_1 - i\beta\phi_1)\,\partial_-(b\Phi_1 + i\beta\phi_1)\,e^{-\frac{\Phi_2+\Phi_1}{b}}\Big) +$$
$$+\Lambda_2\partial_+(b\Phi_2 + i\beta\phi_2)\,\partial_-(b\Phi_2 - i\beta\phi_2)\,e^{\frac{\Phi_2}{b}-\frac{i\beta}{b}\phi_3} + \Lambda_3 e^{\frac{i\beta}{b}\phi_3} + (\text{counterterms}) + \dots,$$

Restoring the deformed OSp(5|2) sigma model in the UV limit

- Then we fermionize the ϕ_3 field and add the counterterms appearing because of the corrections, coming from the fermionic loops. This after the integrations over the Ψ_1 and $\Psi_2^\dagger$ components yields the following action

$$\mathcal{L} = \frac{1}{2\pi}\left(\sum_{i=1}^{2}(\partial_+\Phi_i)(\partial_-\Phi_i)+\sum_{j=1}^{2}(\partial_+\phi_j)(\partial_-\phi_j)\right)+$$
$$+\Lambda_1\Big(\partial_+(b\Phi_1+i\beta\phi_1)\,\partial_-(b\Phi_1-i\beta\phi_1)\,e^{\frac{\Phi_1-\Phi_2}{b}}+$$
$$+\partial_+(b\Phi_1-i\beta\phi_1)\,\partial_-(b\Phi_1+i\beta\phi_1)\,e^{-\frac{\Phi_2+\Phi_1}{b}}\Big)-$$
$$-i\Lambda_2\,\partial_+(b\Phi_2+i\beta\phi_2)\,\partial_-(b\Phi_2-i\beta\phi_2)\,\psi_1^\dagger\psi_2\,e^{\frac{\Phi_2}{b}}+$$
$$+\frac{4i}{\Lambda_3}\partial_+\psi_2\partial_-\psi_1^\dagger+\frac{8\pi}{\beta^2\Lambda_3^2}\psi_1^\dagger\psi_2\partial_+\psi_2\partial_-\psi_1^\dagger+\Lambda_2\,\partial_+(b\Phi_2+i\beta\phi_2)\,\partial_-(b\Phi_2-i\beta\phi_2)\,e^{\frac{\Phi_2}{b}}+\ldots,$$

- Upon identifying $\Phi_{1,2}=2b x_{2,1}$, $\phi_{1,2}=2b\,\varphi_{2,1}$ and $\Psi_1^\dagger=b\Theta^*$, $\Psi_2=b\Theta$ together with taking the limit $b\to\infty$ and adjusting properly the coefficients $\Lambda_{1,2,3}$ $(\alpha'=\frac{2}{b^2})$ we obtain

$$\mathcal{L}=\frac{1}{4\pi\alpha'}\left(\left(\sum_{i=1}^{2}(\partial_+x_i)(\partial_-x_i)+\sum_{j=1}^{2}(\partial_+\varphi_j)(\partial_-\varphi_j)+i\,(1-i\Theta\Theta^*)\,\partial_+\Theta\partial_-\Theta^*\right)-\right.$$
$$-\Lambda\left(\partial_+(x_2+i\varphi_2)\,\partial_-(x_2-i\varphi_2)\,e^{2x_2-2x_1}+\partial_+(x_2-i\varphi_2)\,\partial_-(x_2+i\varphi_2)\,e^{-2x_2-2x_1}+\right.$$
$$\left.\left.+\partial_+(x_1+i\varphi_1)\,\partial_-(x_1-i\varphi_1)\left(\frac{1}{2}+i\Theta\Theta^*\right)e^{2x_1}\right)+\ldots\right)+\mathcal{O}(\alpha'^0)\,.$$

Conclusions and outlook

- We found the action of the η-deformed $OSp(N|2m)$ sigma models for several N and m and put forward the hypothesis how to generate this action for general N and m.
- The 1-loop RG flow of such models was studied and we found the UV stable solutions. We considered the scaling limit of the deformed $OSp(5|2)$ sigma model action as an example.
- The system of screening charges, which determine the integrable structure of the $OSp(N|2)$ sigma model was built.
- By using it we demonstrated how to restore the sigma model action in the deep UV in the case of $OSp(5|2)$.
- Utilizing our system of screenings to write the dual model with the Toda type interactions we can reproduce the expansion of the S-matrix in the vicinity of the special point $\lambda = \frac{1}{2}$ (work in progress).
- The next interesting step would be to try to adapt the dual description for the sigma models with the non-compact target space (Basso, Zhong'18).

Thanks for your attention!